

Action Minimizing Solutions of The One-Dimensional N -Body Problem With Equal Masses *

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Abstract. When we use variational methods to study the Newtonian N -body problem, the main problem is how to avoid collisions. C. Marchal got a remarkable result, that is, a path minimizing the Lagrangian action functional between two given configurations is always a true (collision-free) solution, so long as the dimension d of physical space \mathbb{R}^d satisfies $d \geq 2$. But Marchal's idea can't apply to the case of the one-dimensional physical space. In this paper, we will study the fixed-ends problem for the one-dimensional Newtonian N -body problem with equal masses to supplement Marchal's result. More precisely, we first get the isolated property of collision moments for a path minimizing the action functional between two given configurations, then, if the particles at two endpoints have the same order, the path minimizing the action functional is always a true (collision-free) solution; otherwise, although there must be collisions for any path, we can prove that there are at most $N! - 1$ collisions for any action minimizing path.

Key Words: N -body problem; Collisions; Variational methods; Central configurations; The fixed-ends problem.

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1 Introduction and Main Results

In Euclidean space \mathbb{R}^d , we consider $N \geq 2$ particles with positive masses, affected by their gravitational interactions. The equation of motion of the N -body problem is written as

$$m_k \ddot{q}_k = \sum_{1 \leq j \leq N, j \neq k} \frac{m_j m_k (q_j - q_k)}{|q_j - q_k|^3}. \quad (1.1)$$

where m_k is the mass and q_k the position of the k -th body. Since these equations are invariant by translation, we can assume that the center of masses is at the origin. Firstly, we set some notations and describe preliminary results that will be needed later. Let \mathcal{X}_d denote the space of configurations for N point particles in Euclidean space \mathbb{R}^d with dimension d , whose center of masses is at the origin, that is, $\mathcal{X}_d = \{q = (q_1, \dots, q_N) \in (\mathbb{R}^d)^N : \sum_{k=1}^N m_k q_k = 0\}$. For each pair of indices $j, k \in \{1, \dots, N\}$, let $\Delta_{(j,k)}$ denote the collision set of the j -th and k -th particles $\Delta_{(j,k)} = \{q \in \mathcal{X}_d : q_j = q_k\}$. Let $\Delta_d = \bigcup_{j,k} \Delta_{(j,k)}$ be the collision set in \mathcal{X}_d . The space of collision-free configurations $\mathcal{X}_d \setminus \Delta_d$ is denoted by $\hat{\mathcal{X}}_d$. Let \mathbb{T} denote the time interval $[T_1, T_2]$. By the path space Λ , we mean the Sobolev space $\Lambda = H^1(\mathbb{T}, \mathcal{X}_d)$; we denote by $\Lambda(q_i, q_f)$ the space of paths $q(t) \in \Lambda$ beginning in the configuration q_i at the moment T_1 and ending in the configuration q_f at the moment T_2 . For a motion $q(t)$ of the N -body problem, we say there is a collision at time t_0 if, for at least two indices, say j and k , $q_k(t) \rightarrow c_k$, $q_l(t) \rightarrow c_l$ as $t \rightarrow t_0$, and $c_j = c_k$. We now ‘cluster’ the particles according to their limit points, that is, according to which particles are colliding each other. So, let the different limit points be c_1, \dots, c_n , and let $S_k = \{j \in \{1, \dots, N\} : q_j(t) \rightarrow c_k \text{ as } t \rightarrow t_0\}$, $k = 1, \dots, n$. We consider the opposite of the potential energy (force function) defined by

$$U(q) = \sum_{k < j} \frac{m_k m_j}{|q_k - q_j|}. \quad (1.2)$$

The kinetic energy is defined (on the tangent bundle of \mathcal{X}_d) by $K = \sum_{j=1}^N \frac{1}{2} m_j |\dot{q}_j|^2$, the total energy is $E = K - U$ and the Lagrangian is $L(q, \dot{q}) = L = K + U = \sum_j \frac{1}{2} m_j |\dot{q}|^2 + \sum_{k < j} \frac{m_k m_j}{|q_k - q_j|}$. Given the Lagrangian L , the positive definite functional $\mathcal{A} : \Lambda \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{A}(q) = \int_{\mathbb{T}} L(q(t), \dot{q}(t)) dt \quad (1.3)$$

is termed as action functional (or the Lagrangian action).

The action functional \mathcal{A} is of class C^1 on the collision-free space $\hat{\Lambda}(q_i, q_f) \subset \Lambda(q_i, q_f)$. Hence the critical point of \mathcal{A} in $\hat{\Lambda}(q_i, q_f)$ is a classical solution (of class C^2) of Newtonian

equations

$$m_j \ddot{q}_j = \frac{\partial U}{\partial q_j}. \quad (1.4)$$

From the viewpoint of the Least Action Principle, action minimizing solutions of the N -body problem are the most important and the simplest, so it is natural to search for minimizers of the Lagrangian action joining two given configurations in a fixed time. It's worth noticing that a lot of results have been founded by the action minimization methods just in recent years, please see [1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 14, 15, 16, 24, 25, 26] and the references therein. Recently, the interest in this problem has grown considerably due to the discovery of the figure eight solution [9].

Since the potential of the N -body problem is singular at collision configurations, the main problem involved in variational minimizations is that collision could occur for an action minimizer, even if the set of collision times has necessarily zero measure, the system undergoes a collision of two or more bodies, which prevents it from being a true solution. Some techniques are created to overcome the difficulty, ultimately, one got a major advance (essentially due to Christian Marchal) in this subject. More specifically, the advance is the following remarkable theorem [16, 8, 12].

Theorem 1.1 (Marchal) *Given the initial moment T_1 , the final moment T_2 ($T_2 > T_1$) and two corresponding N -body configurations $q_i = (q_{i1}, \dots, q_{iN})$, $q_f = (q_{f1}, \dots, q_{fN})$ in \mathbb{R}^d ($d > 1$), an action minimizing path joining q_i to q_f in time $T_2 - T_1$ is collision-free for $t \in (T_1, T_2)$.*

This theorem, together with the lower semicontinuity of the action, implies in particular that there always exists a collision-free minimizing solution joining two given collision-free N -body configurations in a given time.

The idea of Christian Marchal is to compare the average of the Lagrangian action for local deformations in all possible directions for a local isolated collision with the original Lagrangian action. Roughly speaking, Marchal's idea is as following : let $a = 2$, by $\frac{1}{2\pi} \int_0^{2\pi} (\frac{1}{|a+e\sqrt{-1}\theta|} - \frac{1}{|a|}) < 0$ (i.e., the average of the Lagrangian action on local deformations is smaller than the original Lagrangian action), then there must be some θ satisfying $\frac{1}{|a+e\sqrt{-1}\theta|} < \frac{1}{|a|}$; however, in the case of $d = 1$, we have $\frac{\frac{1}{|a+1|} + \frac{1}{|a-1|}}{2} - \frac{1}{|a|} > 0$ (i.e., the average of Lagrangian action on local deformations is bigger than the original Lagrangian action), so Marchal's idea can't apply to the case of the one-dimensional physical space. In fact, Marchal's method is local, but the fixed-ends problem for the one-dimensional Newtonian N -body problem is a more global problem, since given two collinear configurations,

if the particles at two configurations have different order, then any path joining two given configurations suffers collisions for topological reasons, hence Marchal's theorem does not hold for the one-dimensional physical space. Fortunately, the one-dimensional Newtonian N -body problem has its particular characteristics, in particular, the fact that all collinear central configurations are non-degenerate gives us the other facility. Thus, in this paper, by using a different approach, we will study the fixed-ends (Bolza) problem for the one-dimensional Newtonian N -body problem. More precisely, we will prove that the path minimizing the Lagrangian action functional between two given configurations is always a true (collision-free) solution of the one-dimensional N -body problem, if the particles at two endpoints have the same order, where, we say that the particles at configurations $q_i = (q_{i1}, \dots, q_{iN})$ and $q_f = (q_{f1}, \dots, q_{fN})$ have **the same order** if $q_{ij} - q_{ik} \geq 0 \Leftrightarrow q_{fj} - q_{fk} \geq 0$ for any $j \neq k$, in other words, the relations $q_{ij} > q_{ik}$ and $q_{fj} < q_{fk}$ can't hold for any $j \neq k$ at the same time. In particular, if $q_{j_1} < q_{j_2} < \dots < q_{j_N}$, we call (j_1, j_2, \dots, j_N) is **the order of the configuration** (q_1, q_2, \dots, q_N) . This requirement is necessary, since it is obvious that there must be collisions for any path if the particles at two endpoints have different order.

In this paper, we will study the fixed-ends problem for the one-dimensional Newtonian N -body problem with equal masses. Our main results are the following Propositions.

Theorem 1.2 *Suppose the critical path $q(t)$ of the Lagrangian action for the one-dimensional Newtonian N -body problem has a collision at some moment t_0 , every corresponding colliding cluster S_k has n_k elements. If the collision is isolated at time t_0 for some right neighborhood or left neighborhood of t_0 , then we have the following results for some right neighborhood or left neighborhood of t_0 :*

if $n_k = 1$, that is, the cluster S_k is singleton, the body in the cluster is not in a collision, let $j \in S_k$, then $q_j(t) = q_j(t_0) + \dot{q}_j(t_0)(t - t_0) + o(t - t_0)$;
if $n_k \geq 2$, let $j \in S_k$, then $q_j(t) = q_j(t_0) + s_j(t - t_0)^{\frac{2}{3}} + o((t - t_0)^{\frac{2}{3}})$, where $s_j, j \in S_k$ is a central configuration for the particles corresponding to the colliding cluster S_k .

Remark 1.1 *Our results depend strongly on the fact that all collinear central configurations are non-degenerate.*

Theorem 1.3 *Suppose the action minimizer $q(t)$ of the Lagrangian action for the one-dimensional Newtonian N -body problem with equal masses has a collision at moment*

t_0 , then the collision moment t_0 is isolated, that is, there exists some $\varepsilon > 0$, $q(t)$ is collision-free in $(t_0 - \varepsilon, t_0 + \varepsilon)$ except at time t_0 . Hence there are at most finitely many collision moments for the fixed-ends (Bolza) problem.

Remark 1.2 *There are some studies about the isolated collision for the general N -body problem (see [8, 12, 22]). However, all the results of them only said that: **there exists an isolated collision** for the general N -body problem. Our results show that we can say more about the one-dimensional Newtonian N -body problem with equal masses: all the collisions are isolated and finite.*

Theorem 1.4 *For the one-dimensional N -body problem with equal masses, given the initial moment T_1 , the final moment T_2 ($T_2 > T_1$) and two corresponding N -body configurations $q_i = (q_{i1}, \dots, q_{iN})$, $q_f = (q_{f1}, \dots, q_{fN})$ in \mathbb{R}^1 , if q_i, q_f have the same order in \mathbb{R}^1 , then the action minimizing path of the fixed-ends problem joining q_i to q_f in time $T_2 - T_1$ is collision-free for $t \in (T_1, T_2)$.*

Theorem 1.5 *If the given two configurations q_i, q_f have the different order in \mathbb{R}^1 , then the action minimizing path of the fixed-ends problem with equal masses joining q_i to q_f in time $T_2 - T_1$ has some collisions for some $t \in (T_1, T_2)$, but there are at most $N! - 1$ collision moments in (T_1, T_2) .*

Remark 1.3 *Our results and methods remain valid for more general force function defined by $U(q) = \sum_{k < j} \frac{m_k m_j}{|q_k - q_j|^\alpha}$, where α is any positive real number such that $0 < \alpha < 2$.*

It is natural to ask the following questions.

Question. 1. Do the **Theorem 1.3, 1.4 and 1.5** hold for the one-dimensional N -body problem with any masses? 2. Given two configurations which have the different order in \mathbb{R}^1 and a time $T = T_2 - T_1 > 0$, what is the largest number of collision times in (T_1, T_2) ? Is the largest number of collision times in (T_1, T_2) one? The similar questions can be asked for the fixed-ends problem with any masses.

We hope that the answers of these questions are all positive.

The paper is structured as follows. **Section 2** introduces some definitions and some lemmas, **Section 3** gives the proofs of the main results by using the concepts and results introduced in **Section 1** and **Section 2**.

2 Some Definitions and Some Lemmas

In this section, we give some definitions and recall some classical results.

The first one is the important concept of the central configuration [23],

Definition 2.1 *A configuration $q = (q_1, \dots, q_N) \in \mathcal{X}_d \setminus \Delta_d$ is called a central configuration if there exists a constant $\lambda \in \mathbb{R}$ such that*

$$\sum_{j=1, j \neq k}^N \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) = -\lambda m_k q_k, 1 \leq k \leq N, \quad (2.1)$$

the value of λ in (2.1) is uniquely determined by

$$\lambda = \frac{U(q)}{I(q)}, \quad (2.2)$$

where

$$I(q) = \sum_{1 \leq j \leq N} m_j |q_j|^2. \quad (2.3)$$

Let us recall that, for a motion $q(t)$ of N -body problem, we say there is a collision at time t_0 if as $t \rightarrow t_0$, $q_j(t) \rightarrow c_j$, $j \in \{1, \dots, N\}$ and for at least two different indices, say j and k such that $c_j = c_k$. Without loss of generality, we can assume that the time t approach t_0 from the right of t_0 , that is, we think $t \rightarrow t_0+$. Denote the different limit points by c_1, \dots, c_n , and classify the indices according to particles colliding each other, let $S_k = \{j \in \{1, \dots, N\} : q_j(t) \rightarrow c_k \text{ as } t \rightarrow t_0+\}$, and assume S_k has n_k elements for $k = 1, \dots, n$; then we say that every S_k is a **colliding cluster** of particles. Let $M_k = \sum_{j \in S_k} m_j$ be the total mass of particles in cluster S_k and $\bar{c}_k = \sum_{j \in S_k} m_j q_j / M_k$ be the center of mass of the particles in S_k .

When S_k has $n_k \geq 2$ elements, if $j \in S_k$, let $r_{(k)j}(t) = \frac{q_j - c_k}{(t - t_0)^{\frac{2}{3}}}$, then we call $r_{(k)}(t) = (r_{(k)l_1}(t), \dots, r_{(k)l_{n_k}}(t))$ be the **normalized configuration** corresponding to the colliding cluster S_k , where $\{l_1, \dots, l_{n_k}\} = S_k$. Let

$$\mathbf{CC}_k := \{r_{(k)} : \sum_{j \in S_k, j \neq i} \frac{m_j}{|r_{(k)j} - r_{(k)i}|^3} (r_{(k)j} - r_{(k)i}) = -\frac{2}{9} r_{(k)i}, i \in S_k\} \quad (2.4)$$

be the set of the central configuration corresponding to colliding cluster S_k , where we assume the value of λ which only affects the size of the central configuration to be $\frac{2}{9}$, note that the center of mass of $r_{(k)}$ is zero.

Before giving the proofs of the main results of this paper, some lemmas are needed. we recall some classical results concerning a motion $q(t)$ of N -body problem in some neighborhood of isolated collision instant t_0 .

The first one says that all collision orbits of N -body problem in some neighborhood of isolated collision instant t_0 have the property that $r_{(k)}(t) \rightarrow \mathbf{CC}_k$ as $t \rightarrow t_0$, where $r_{(k)}(t)$ and \mathbf{CC}_k are respectively the normalized configuration of the collision orbit and the set of the central configuration corresponding to colliding cluster S_k .

Lemma 2.1 *Suppose a colliding cluster S_k have $n_k \geq 2$ elements, let $r_j(t) = \frac{q_j - c_k}{(t - t_0)^{\frac{2}{3}}}$ for any $j \in S_k$, be the **normalized configuration**. Then for every converging sequence $r(t_j) = (r_{l_1}(t_j), \dots, r_{l_{n_k}}(t_j))$, where $l_1, \dots, l_{n_k} \in S_k$, t_j belong to some neighborhood of t_0 ($j \in \mathbb{N}$), the limit $\lim_{j \rightarrow \infty} r(t_j) := s$ is a central configuration.*

Remark 2.1 *This result is classical(see [18, 12] for a proof). Because of the called (Painlevé-Wintner) infinite spin problem(see [23, 19, 18, 5, 8] et al), in general, one can not get a better result.*

The second one states the special property, which we need, of the one-dimensional Newtonian N -body problem.

Lemma 2.2 ([17]) *All collinear central configurations are non-degenerate in \mathbb{R}^d .*

Then, in the following, we get the important result which says that, for a isolated collision of particles, not only does $r_{(k)}(t) \rightarrow \mathbf{CC}_k$ as $t \rightarrow t_0$, but also there is a central configuration $s \in \mathbf{CC}_k$ so that $r_{(k)}(t) \rightarrow s$ as $t \rightarrow t_0$, so long as all central configurations are non-degenerate.

Lemma 2.3 *For the one-dimensional N -body problem, suppose a colliding cluster S_k have $n_k \geq 2$ elements, let $r_j(t) = \frac{q_j - c_k}{(t - t_0)^{\frac{2}{3}}}$ for any $j \in S_k$, be the normalized configuration. Then $\lim_{t \rightarrow t_0} r(t)$ exists, the limit $s := \lim_{t \rightarrow t_0} r(t)$ is a central configuration, furthermore, s and $r(t)$ have the same order.*

Proof of Lemma 2.3:

It's similar to a particular case of the results of Saari [18], we can get **lemma 2.3** by using the unstable manifold theorem for a normally hyperbolic invariant set (Hirsch et al. [13]) and **Lemma 2.2**.

□

Remark 2.2 *There are some methods to study this important problem(see [23, 20, 21, 19, 18, 11, 5, 8, 12]et al). To our knowledge, **Lemma 2.3** was not definitely stated. Since all collinear central configurations are non-degenerate, we apply the idea of D.Saari (the unstable manifold theorem for a normally hyperbolic invariant set) to simply get the result.*

The last lemma is about the existence of isolated collisions for the general N -body problem.

Lemma 2.4 ([8, 12]) *Suppose the action minimizer $q(t)$ of the Newtonian N -body problem has collisions in a time interval, then there must exist an isolated collision in this time interval.*

Using above lemmas, we will give the proofs of our main results in the next section.

3 The Proofs of Main Results

In this section, we give the proofs of main results in this paper.

Proof of Theorem 1.2:

This result easily comes from **Lemma 2.3**.

□

First of all, let's establish a lemma to simplify the proofs of other theorems.

Lemma 3.1 *Given the initial moment T_1 , the final moment T_2 ($T_2 > T_1$) and two corresponding N -body configurations $q_i = (q_{i1}, \dots, q_{iN})$, $q_f = (q_{f1}, \dots, q_{fN}) \in \mathcal{X}_1 \setminus \Delta_1$ which have the same order in \mathbb{R}^1 . Suppose a path $q(t) \in \Lambda(q_i, q_f)$ has only one collision moment t_0 in (T_1, T_2) , then the path $q(t)$ cannot be an action minimizing path of the fixed-ends problem joining q_i to q_f in time $T_2 - T_1$.*

Proof of Lemma 3.1:

By using reduction to absurdity, assume that the path $q(t)$ is an action minimizing path of the fixed-ends problem joining q_i to q_f in time $T_2 - T_1$. Without loss of generality, we can assume that $q_1(t) < q_2(t) < \dots < q_N(t)$ for $t \in [T_1, T_2] \setminus \{t_0\}$ and $q_1(t_0) \leq q_2(t_0) \leq \dots \leq q_N(t_0)$.

Let $x_k(t) = q_{k+1}(t) - q_k(t)$ for $k \in \{1, \dots, N-1\}$ and $M = m_1 + m_2 + \dots + m_N$, then $x(t) = (x_1(t), x_2(t), \dots, x_{N-1}(t))$ is an action minimizing path of the fixed-ends problem joining $x_i = x(T_1)$ to $x_f = x(T_2)$ in time $T_2 - T_1$ for the action functional

$$\mathcal{F}(x) = \int_{\mathbb{T}} \sum_{1 \leq l < k \leq N} \frac{m_k m_l}{2M} \left[\sum_{l \leq j \leq k-1} \dot{x}_j|^2 + \frac{2M}{|\sum_{l \leq j \leq k-1} x_j|} \right] dt \quad (3.1)$$

In fact, by Lagrangian identity, we have

$$\begin{aligned} \mathcal{A}(q) &= \int_{\mathbb{T}} L(q(t), \dot{q}(t)) dt \\ &= \int_{\mathbb{T}} \left[\frac{1}{2(\sum_{1 \leq j \leq N} m_j)} \sum_{1 \leq l < k \leq N} m_k m_l |\dot{q}_k - \dot{q}_l|^2 + \sum_{1 \leq l < k \leq N} \frac{m_k m_l}{|q_k - q_l|} \right] dt \\ &= \int_{\mathbb{T}} \sum_{1 \leq l < k \leq N} \frac{m_k m_l}{2M} \left[\sum_{l \leq j \leq k-1} \dot{x}_j|^2 + \frac{2M}{|\sum_{l \leq j \leq k-1} x_j|} \right] dt \\ &= \mathcal{F}(x) \end{aligned}$$

In the following, we will construct another path $y(t)$ which satisfies the same boundary conditions with $x(t)$, but the value of $\mathcal{F}(y)$ is smaller than the value of $\mathcal{F}(x)$.

Since we can get similar result by using the following method for any $k \geq 1$ such that $x_k(t) \rightarrow 0$ when $t \rightarrow t_0$, for the sake of convenience, we only consider that $x_1(t) \rightarrow 0$ when $t \rightarrow t_0$. Then we have $x_1(t) = \alpha(t_0 - t)^{\frac{2}{3}} + o((t_0 - t)^{\frac{2}{3}})$ for some left neighborhood of t_0 and $x_1(t) = \beta(t - t_0)^{\frac{2}{3}} + o((t - t_0)^{\frac{2}{3}})$ for some right neighborhood of t_0 from **Theorem 1.2**, where α, β are appropriate positive numbers. Let $A = \frac{m_1(M-m_1)}{2M}$ and $B = \sum_{3 \leq k \leq N} \frac{m_1 m_k}{M} \sum_{2 \leq j \leq k-1} \dot{x}_j$, from **Theorem 1.2** we know that

- if $x_j(t) \rightarrow 0$ when $t \rightarrow t_1$ for some $j \in \{2, \dots, N-1\}$, then $B = \frac{d(\tilde{\alpha}(t_0-t)^{\frac{2}{3}} + o((t_0-t)^{\frac{2}{3}}))}{dt}$ for some left neighborhood of t_0 and $B = \frac{d(\tilde{\beta}(t-t_0)^{\frac{2}{3}} + o((t-t_0)^{\frac{2}{3}}))}{dt}$ for some right neighborhood of t_0 , where $\tilde{\alpha}, \tilde{\beta}$ are appropriate positive numbers;
- if $x_j(t) > 0$ for some neighborhood of t_0 and any $j \in \{2, \dots, N-1\}$, then $B = \frac{d(a+b(t-t_0)+o(|t_1-t|))}{dt}$ for some neighborhood of t_0 , where $a > 0, b$ are appropriate real numbers.

Then it is easy to know that the inequality

$$A\dot{x}_1^2 + B\dot{x}_1 > 0 \quad (3.2)$$

holds in some neighborhood of t_0 . For sufficiently small positive number δ , there are two sufficiently small positive numbers ϵ, ε such that $x_1(t_1 - \epsilon) = x_1(t_1 + \varepsilon) = \delta$, $x_1(t) \leq \delta$

for $t \in [t_1 - \epsilon, t_1 + \epsilon]$ and the interval $[t_1 - \epsilon, t_1 + \epsilon]$ is in this neighborhood of t_0 for the inequality (3.2) holds. Furthermore, we have the inequalities

$$\frac{1}{|x_1 + \sum_{2 \leq j \leq k-1} x_j|} \geq \frac{1}{|\delta + \sum_{2 \leq j \leq k-1} x_j|} \quad (3.3)$$

for $t \in [t_1 - \epsilon, t_1 + \epsilon]$ and any $3 \leq k \leq N$.

Let $y_1(t) = \delta$ for $t \in [t_1 - \epsilon, t_1 + \epsilon]$, $y_1(t) = x_1(t)$ for $t \in [T_1, T_2] \setminus [t_1 - \epsilon, t_1 + \epsilon]$, and $y_j(t) = x_j(t)$ for $t \in [T_1, T_2]$ and $2 \leq j \leq N - 1$. Let $y(t) = (y_1(t), y_2(t), \dots, y_{N-1}(t))$, then we know

$$\begin{aligned} \mathcal{F}(x) - \mathcal{F}(y) &= \int_{t_1 - \epsilon}^{t_1 + \epsilon} \sum_{1 \leq l < k \leq N} \frac{m_k m_l}{2M} \left[\sum_{l \leq j \leq k-1} \dot{x}_j^2 + \frac{2M}{|\sum_{l \leq j \leq k-1} x_j|} \right] dt \\ &\quad - \int_{t_1 - \epsilon}^{t_1 + \epsilon} \sum_{1 \leq l < k \leq N} \frac{m_k m_l}{2M} \left[\sum_{l \leq j \leq k-1} \dot{y}_j^2 + \frac{2M}{|\sum_{l \leq j \leq k-1} y_j|} \right] dt \\ &= \int_{t_1 - \epsilon}^{t_1 + \epsilon} \left[A \dot{x}_1^2 + B \dot{x}_1 + \sum_{3 \leq k \leq N} \frac{m_k m_1}{|x_1 + \sum_{2 \leq j \leq k-1} x_j|} \right] dt \\ &\quad - \int_{t_1 - \epsilon}^{t_1 + \epsilon} \sum_{3 \leq k \leq N} \frac{m_k m_1}{|\delta + \sum_{2 \leq j \leq k-1} x_j|} dt \\ &> 0 \end{aligned}$$

Hence the path $q(t)$ is not an action minimizing path of the fixed-ends problem joining q_i to q_f in time $T_2 - T_1$. □

Henceforth, we think all the particles have equal mass, i.e., we assume $m_1 = m_2 = \dots = m_N = m$.

Proof of Theorem 1.3:

By using reduction to absurdity, without loss of generality, let t_0 be an instant at which collision times accumulate for some right neighborhood of t_0 . By **Lemma 2.4**, there are infinite isolated collisions in some right neighborhood of t_0 . Then it's easy to know that there are three isolated collision moments t_1, t_2 and t_3 ($t_1 < t_2 < t_3$) such that the collisions at moments t_1, t_2 and t_3 have the same colliding clusters and the same order, i.e., as $t \rightarrow t_i$ ($i \in \{1, 2, 3\}$), there exist different limit points c_{i1}, \dots, c_{in} such that $S_{ik} = \{j \in \{1, \dots, N\} : q_j(t) \rightarrow c_{ik} \text{ as } t \rightarrow t_i\}$ and $S_{1k} = S_{2k} = S_{3k}$ for $k = 1, \dots, n$, furthermore, (without loss of generality) $c_{i1} < \dots < c_{in}$ for $i \in \{1, 2, 3\}$. Given $k \in \{1, \dots, n\}$, if the colliding cluster S_{2k} has $n_k \geq 2$ elements, suppose the order of the particles in S_{2k} is (l_1, \dots, l_{n_k}) for some left neighborhood of t_2 and (j_1, \dots, j_{n_k}) for

some right neighborhood of t_2 , that is, $q_{l_1}(t) < \dots < q_{l_{n_k}}(t)$ for some left neighborhood of t_2 and $q_{j_1}(t) < \dots < q_{j_{n_k}}(t)$ for some right neighborhood of t_2 , where $\{l_1, \dots, l_{n_k}\} = \{j_1, \dots, j_{n_k}\} = S_{2k}$. If $(l_1, \dots, l_{n_k}) \neq (j_1, \dots, j_{n_k})$, assume τ_k is a permutation from (l_1, \dots, l_{n_k}) to (j_1, \dots, j_{n_k}) , let

$$(h_{j_1}(t), \dots, h_{j_{n_k}}(t)) = (q_{\tau_k(l_1)}(t), \dots, q_{\tau_k(l_{n_k})}(t)) \quad (3.4)$$

for $t \in [t_1, t_2]$. If $(l_1, \dots, l_{n_k}) = (j_1, \dots, j_{n_k})$, or if the colliding cluster S_{2k} has $n_k = 1$ element, that is, the cluster S_{2k} is singleton, thus the body in the cluster is not in a collision, the permutation τ_k can be chosen as unit transformation, then still let

$$(h_{j_1}(t), \dots, h_{j_{n_k}}(t)) = (q_{\tau_k(l_1)}(t), \dots, q_{\tau_k(l_{n_k})}(t)) \quad (3.5)$$

for $t \in [t_1, t_2]$.

Finally, let $h(t) = (h_1(t), \dots, h_N(t))$ for $t \in [t_1, t_2]$ and $h(t) = (q_1(t), \dots, q_N(t))$ for $t \in [t_2, t_3]$, then $h(t)$ is a path in the Sobolev space $H^1([t_1, t_3], \mathcal{X}_1)$ with fixed-ends such that $h(t_1) = q(t_1)$ and $h(t_3) = q(t_3)$. Indeed, by the construction of $h(t)$, the relations $h(t_1) = q(t_1)$ and $h(t_3) = q(t_3)$ are obvious; by the continuity of $h(t)$ at $t = t_2$, it's easy to know that $h(t)$ has weak derivative $\dot{h}(t)$ in $[t_1, t_3]$, furthermore, $\dot{h}(t)$ is square integrable in $[t_1, t_3]$ by applying the finiteness of the Lagrangian action.

Let us recall that, if all the particles have the same masses, there is an obvious fact: *suppose τ is a permutation of $(1, 2, \dots, N)$, let $r(t) = (r_1(t), r_2(t), \dots, r_N(t)) = (q_{\tau(1)}(t), q_{\tau(2)}(t), \dots, q_{\tau(N)}(t))$, if $m_1 = m_2 = \dots = m_N$, then*

$$\int_{T_1}^{T_2} L(q(t), \dot{q}(t)) dt = \int_{T_1}^{T_2} L(r(t), \dot{r}(t)) dt, \quad (3.6)$$

Since the path $q(t)$ is an action minimizing path, we know that the path $h(t)$ is an action minimizing path in the Sobolev space $H^1([t_1, t_3], \mathcal{X}_1)$ with fixed-ends $h(t_1) = q(t_1)$ and $h(t_3) = q(t_3)$. In particular, the path $h(t)$ is an action minimizing path in the Sobolev space $H^1([t_2 - \epsilon, t_2 + \epsilon], \mathcal{X}_1)$ with fixed-ends $h(t_2 - \epsilon)$ and $h(t_2 + \epsilon)$ for all the sufficiently small $\epsilon > 0$. By choosing any sufficiently small $\epsilon > 0$, we have a path $h(t)$ such that: the action minimizing path $h(t) \in H^1([t_2 - \epsilon, t_2 + \epsilon], \mathcal{X}_1)$ has only one collision moment t_2 in $(t_2 - \epsilon, t_2 + \epsilon)$, the fixed-ends $h(t_2 - \epsilon), h(t_2 + \epsilon) \in \mathcal{X}_1 \setminus \Delta_1$ and have the same order in \mathbb{R}^1 . However, this contradicts with **Lemma 3.1**.

In conclusion, if the action minimizing path $q(t)$ of the one-dimensional Newtonian N -body problem with equal masses has collisions, then every collision is isolated. Since

the set of collision times is closed, we know there are at most finitely many collision moments for the fixed-ends (Bolza) problem.

□

Proof of Theorem 1.4:

First of all, let's establish a lemma to simplify the proof.

Lemma 3.2 *Given the initial moment T_1 , the final moment T_2 ($T_2 > T_1$) and two corresponding N -body configurations $q_i = (q_{i1}, \dots, q_{iN})$, $q_f = (q_{f1}, \dots, q_{fN})$ which have the same order in \mathbb{R}^1 , suppose the path $q(t) \in \Lambda(q_i, q_f)$ has collision in (T_1, T_2) , and the collision moments in (T_1, T_2) are respectively t_1, t_2, \dots, t_n ($T_1 < t_1 < \dots < t_n < T_2$). Then there is some path $h(t) \in \Lambda(q_i, q_f)$ such that $\{t_1, \dots, t_n\}$ are collision moments in (T_1, T_2) and the order of $h(t)$ are the same for all the time $t \in [T_1, T_2]$. Furthermore, if all the particles have the same masses, then*

$$\int_{T_1}^{T_2} L(q(t), \dot{q}(t)) dt = \int_{T_1}^{T_2} L(h(t), \dot{h}(t)) dt. \quad (3.7)$$

Proof of Lemma 3.2:

It's easy to know that, there is some path $g(t)$ which has the same order with q_i and q_f in \mathbb{R}^1 and $g(t)$ is collision-free for $t \in (T_1, T_2)$. Suppose the order of the orbit $g(t)$ for $t \in (T_1, T_2)$ is (j_1, \dots, j_N) , that is, $g_{j_1}(t) < \dots < g_{j_N}(t)$, where $\{j_1, \dots, j_N\} = \{1, \dots, N\}$. Without loss of generality, we can assume that $(j_1, j_2, \dots, j_N) = (1, 2, \dots, N)$. Let $t_0 = T_1$ and $t_{n+1} = T_2$, suppose the order of the orbit $q(t)$ for $t \in (t_k, t_{k+1})$ is (j_{k1}, \dots, j_{kN}) , that is, $q_{j_{k1}}(t) < \dots < q_{j_{kN}}(t)$, where $k \in \{0, \dots, n\}$. Suppose τ_k is a permutation from (j_{k1}, \dots, j_{kN}) to $(1, 2, \dots, N)$, let

$$h^{(k)}(t) = (h_1^{(k)}(t), h_2^{(k)}(t), \dots, h_N^{(k)}(t)) = (q_{\tau_k(1)}(t), q_{\tau_k(2)}(t), \dots, q_{\tau_k(N)}(t)) \quad (3.8)$$

for $t \in (t_k, t_{k+1})$. Firstly, it is easy to know that

$$\lim_{t \rightarrow t_0^+} h^{(0)}(t) = q_i, \quad \lim_{t \rightarrow t_{n+1}^-} h^{(n)}(t) = q_f \quad (3.9)$$

In the following, we prove that

$$\lim_{t \rightarrow t_{k+1}^-} h_j^{(k)}(t) = \lim_{t \rightarrow t_{k+1}^+} h_j^{(k+1)}(t) \quad (3.10)$$

for every $j \in \{1, \dots, N\}$ and $k \in \{0, \dots, n-1\}$.

In fact, from $h_j^{(k)}(t) = q_{\tau_k(j)}(t)$ for $t \in (t_k, t_{k+1})$ and $h_j^{(k+1)}(t) = q_{\tau_{k+1}(j)}(t)$ for $t \in (t_{k+1}, t_{k+2})$, it is easy to know that we only need to prove the relation $q_{\tau_k(j)}(t_{k+1}) = q_{\tau_{k+1}(j)}(t_{k+1})$. For the sake of a contradiction, we can suppose that $q_{\tau_k(j)}(t_{k+1}) > q_{\tau_{k+1}(j)}(t_{k+1})$ or $q_{\tau_k(j)}(t_{k+1}) < q_{\tau_{k+1}(j)}(t_{k+1})$. If $q_{\tau_k(j)}(t_{k+1}) > q_{\tau_{k+1}(j)}(t_{k+1})$, from $h_l^{(k)}(t) > h_j^{(k)}(t)$ for $N \geq l > j, t \in (t_k, t_{k+1})$, we have $q_{\tau_k(l)}(t_{k+1}) = \lim_{t \rightarrow t_{k+1}^-} h_l^{(k)}(t) \geq \lim_{t \rightarrow t_{k+1}^-} h_j^{(k)}(t) = q_{\tau_k(j)}(t_{k+1}) > q_{\tau_{k+1}(j)}(t_{k+1})$. Hence $h_{\tau_{k+1}^{-1}\tau_k(l)}^{(k+1)}(t) = q_{\tau_k(l)}(t) > q_{\tau_{k+1}(j)}(t) = h_j^{(k+1)}(t)$ for every l such that $N \geq l \geq j, t \in (t_{k+1}, t_{k+1} + \epsilon)$, where ϵ is some sufficiently small positive number. So we have $\tau_{k+1}^{-1}\tau_k(l) > j$ for every l such that $N \geq l \geq j$, but there are at most $N - j$ number larger than j in $\{1, 2, \dots, N\}$, this is a contradiction. If $q_{\tau_k(j)}(t_{k+1}) < q_{\tau_{k+1}(j)}(t_{k+1})$, it is similar to get a contradiction. So we have

$$\lim_{t \rightarrow t_{k+1}^-} h_j^{(k)}(t) = \lim_{t \rightarrow t_{k+1}^+} h_j^{(k+1)}(t) \quad (3.11)$$

for every $j \in \{1, \dots, N\}$ and $k \in \{0, \dots, n-1\}$.

Let $h(t) = h^{(k)}(t)$ for $t \in (t_k, t_{k+1})$, $h(t_k) = \lim_{t \rightarrow t_k^+} h^{(k)}(t)$ for $1 \leq k \leq n$, $h(T_1) = \lim_{t \rightarrow t_0^+} h^{(0)}(t)$, $h(T_2) = \lim_{t \rightarrow t_{n+1}^-} h^{(n)}(t)$, then $h(t) \in \Lambda(q_i, q_f)$ and $\{t_1, \dots, t_n\}$ are collision moments in (T_1, T_2) and the order of $h(t)$ are the same for all the time $t \in [T_1, T_2]$.

Furthermore, since all the particles have the same masses, we have

$$\int_{T_1}^{T_2} L(q(t), \dot{q}(t)) dt = \int_{T_1}^{T_2} L(h(t), \dot{h}(t)) dt. \quad (3.12)$$

From the above, **Lemma 3.2** holds.

In the following, we prove **Theorem 1.4** by using **Lemma 3.2**.

By using reduction to absurdity, suppose the action minimizing path $q(t)$ has collision moments in (T_1, T_2) , the collision moments in (T_1, T_2) are respectively t_1, t_2, \dots, t_n ($T_1 < t_1 < \dots < t_n < T_2$). Furthermore, we can assume that $q_1(t) < q_2(t) < \dots < q_N(t)$ for $t \in (T_1, T_2) \setminus \{t_1, \dots, t_n\}$ and $q_1(t_k) \leq q_2(t_k) \leq \dots \leq q_N(t_k)$ for $k \in \{1, \dots, n\}$ by using **Lemma 3.2**. Then we can find a path $q(t) \in H^1([t_1 - \epsilon, t_1 + \epsilon], \mathcal{X}_1)$ which has only one collision moment t_1 in $(t_1 - \epsilon, t_1 + \epsilon)$, and the fixed-ends $q(t_1 - \epsilon), q(t_1 + \epsilon) \in \mathcal{X}_1 \setminus \Delta_1$ have the same order in \mathbb{R}^1 , so long as the positive number ϵ is sufficiently small. However, this contradicts with **Lemma 3.1**.

So we know that, for the N-body problem with equal masses, given two moments and corresponding configurations which have the same order in \mathbb{R}^1 , the action minimizing path of the fixed-ends problem joining two configurations is collision-free for $t \in (T_1, T_2)$. \square

Proof of Theorem 1.5:

Suppose the action minimizing orbit $q(t)$ has collision in (T_1, T_2) , the collision moments in (T_1, T_2) are respectively t_1, t_2, \dots, t_n ($T_1 < t_1 < \dots < t_n < T_2$), let $t_0 = T_1$ and $t_{n+1} = T_2$. Let us investigate $n + 1$ collision-free path sections: $q(t), t \in (t_k, t_{k+1})$, $0 \leq k \leq n$. If $n > N! - 1$, then there are two sections which have the same order, suppose the corresponding time intervals are respectively (t_j, t_{j+1}) and (t_l, t_{l+1}) , $j < l$. Let us choose two moments $s_1 \in (t_j, t_{j+1})$ and $s_2 \in (t_l, t_{l+1})$, then it is easy to know that the path $q(t), t \in [s_1, s_2]$ is an action minimizing orbit of the fixed-ends problem for two moments s_1, s_2 and corresponding configurations $q(s_1), q(s_2)$. However, from **Theorem 1.4**, $q(t)$ is collision-free in (s_1, s_2) , this contradicts with $t_{j+1}, t_l \in (s_1, s_2)$.

□

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